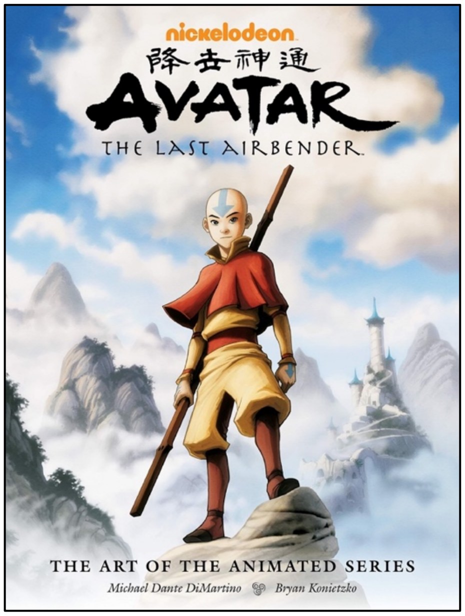
Avatar: The Last Airbender



Avatar: The Last Airbender[1]

When my math teacher asked me what my favorite TV show was, I immediately thought of Avatar: The Last Airbender. It was the cartoon that took me through my childhood and taught me endless life philosophies. I've finished through all 61 episodes more times than I can count and am able to clearly recall details of each episode. When realizing the hidden mathematics behind every second of it all, I had to delve deeper.

In short, Avatar: The Last Airbender takes place in a world where there are four nations (Earth, Water, Fire, and Air).[2] In these nations, respectively, there are benders of that element.[3] A bender is someone who has the ability to control an element of life from the four listed above.

[1] Konietzko, Bryan, et al. *Avatar, the Last Airbender*. Dark Horse Books, 2011.

[2] "The Last Airbender." *IMDb*, IMDb.com, 2010.

[3] Ibid.

You'd find water-benders in the Water nation, earth-benders in the Earth nation, and so on. While being a bender is useful in everyday life---like moving rocks to save people stuck or controlling ocean waters to move a boat---benders also use their abilities when fighting.

In season 2 episode 6 of Avatar, the main characters (Aang, Katara, and Sokka) go to a tournament in the Earth nation where earth-benders fight each other for a cash prize.[4] While most fighters seem like amateurs, Aang notices one particular earth-bender... The Blind Bandit. The Blind Bandit, or Toph, is a small, blind girl who uses her advanced earth-bending skills to listen closely and feel the movements around her, in order to know where, when, and how to attack.

Toph[5]

The aim of this paper is to create a model for Toph's battle in the arena, analyzing how she prepares and counters against her opponents and their attacks through her earth-bending. This will be done mostly through the use of graphing, functions, and equations, with an explanation of key features, transformations, quadratic and linear equations, as well as solving both graphically and analytically.

[4] "Avatar: The Last Airbender." *Metacritic*, 5 May 2006.

[5] "Toph Beifong." *Avatar Wiki*, avatar.fandom.com/wiki/Toph_Beifong.

The Situation

A number of fighters just went head-to-head in a tournament with the best earth-benders in town. All of the fighters, shown below, would attempt to get to the final round to fight the Blind Bandit (the reigning champ) for the championship belt. The Boulder, on the far left of the image, has just made it to the end and is now face-to-face with Toph.

Toph, as previously mentioned, is blind. Aang, the Avatar and main character, is currently looking for an earth-bending teacher who, according to a revelation, "waits and listens."[1] He pays special attention to how the Blind Bandit fights.

So how does she fight?

Earth-benders at the Tournament[6]

If you create the function of the opponent's movements, then graph it on top of the function of Toph's motions and attacks, it should not only give us an answer as to how Toph fights, but also a visual representation of the fight itself and an intersection point which represents exactly when the attack will land.

[6] Konietzko, Bryan, and Michael Dante DiMartino. "Avatar: The Last Airbender."

First, it is necessary to assess variables and consider what functions may be used. For the opponent function, movement can be graphed in a number of ways. We could measure his speed over time or his distance from Toph, to mention some. However, considering the aim (to not only graph the battle, but provide a visual representation with the graph), the best variables for the opponent function will be x, in time passed, and y, in feet above ground, to represent the changing altitude of the opponent's foot over time.

x (in elapsed time) y (in feet above ground)

It's likely that the movement of the opponent will be at a constant rate, so a linear function would fit best here. The basic function can be described as:

$f(x) = mx + b,$ where y is defined by f(x), m is the slope, and b is the y-intercept

The slope is defined by change in y over change in x, while the y-intercept is the y-value when x equals zero.

Toph's function, on the other hand, will have to be exponential, including somewhat of a j-curve. The rationale for this is simply because a straightforward attack at a constant rate is predictable. It won't work. For the equation to fit the scenario, it will have to build slowly, then attack quickly.

The basic quadratic function can be defined as:

$f(x) = ax^2 + bx + c$, where coefficients a, b, and c describe the function

The first term, seen above, is the quadratic term (the coefficient for the x being squared), where a determines the shape (being wide/narrow) and direction (being positive/negative) of the graph.[7]

The middle term, with coefficient b, is the linear term, which is used to alter the axis of symmetry.[8]

The last term, c, is the y-intercept (the y-value when x equals 0), which is useful for determining the starting point of a graph.[9]

Creating an Opponent Function

The first step is to form The Boulder's function, using the basic function $f(x) = mx + b$. The actions look something like this:

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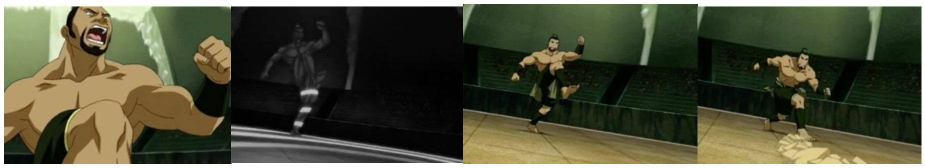
[7] "Solving and Graphing Quadratic Equations." *American Board for Certification of Teacher Excellence.*

[8] Ibid.

[9] Ibid.



Lifting and planting foot[10]

For the series of events, one linear equation will not be able to accurately describe what is going on. Therefore, multiple windows will be necessary. Since there are four main stages, (grounded foot, upwards movement, downwards movement, then planted foot), we can divide the graph into four sections (color coded below).

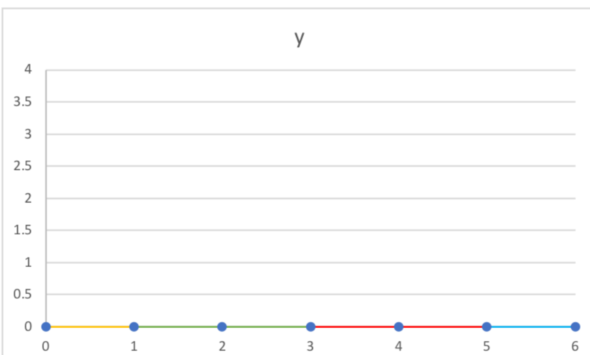


Figure 1.1

For the labels, The Boulder's function will be measured by time elapsed (x-axis) and distance of the foot from the ground (y-axis), which gives us a visual of his physical position at all times.

Figure 1.2

[10] Konietzko, Bryan, and Michael Dante DiMartino. "Avatar: The Last Airbender."

The Boulder is a hefty guy, but he doesn't look too slow either, so the windows will be made at 1 second intervals. The first and fourth function, since there is net-zero movement, are defined by the equation $f(x) = 0$, where they will remain on the x-axis. Their domains are $0 \leq x < 1$ and $3 \leq x < 4$, respectively.

The second function, in the window of 1 to 2 seconds, will represent the knee going up. Assuming that his foot reaches a maximum height of 3 feet, which is me guessing from how it looks in the episode, the function has to pass through the points (1, 0) and (2,3).

Using the slope formula (change in y over change in x), we see that the slope of this function is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{3-0}{2-1} = \frac{3}{1} = 3$$

Then, by plugging in a point with the slope into the basic function, the y-intercept b is:

$$f(x) = mx + b$$

$$3 = (3)(2) + b$$

$$3 = (6) + b$$

$$b = -3$$

Thus,

$$f(x) = 3x - 3, \text{ for } 1 \leq x < 2$$

On the other hand, the third function goes through the points (3, 3) and (4, 0). Using the same steps as above, the slope for this function will be -3. When we plug -3 in for m, we see that the y-intercept is:

$$3 = (-3)(2) + b$$

$$3 = (-6) + b$$

$$b = 9$$

Hence,

$$f(x) = -3x + 9, \text{ for } 2 \leq x < 3$$

When we put these four equations together, the graph looks like this:

Figure 1

Putting these points into a graph reveals...

x	0	1	2	3	4
y	0	0	3	0	0

Table 1.1

Well, not much. But adding more points with smaller intervals may create more accurate depiction of what is going on in the graph.

x	y
0	0
...	...
1	0
1.25	.75
1.5	1.5
1.75	2.25
2	3
2.25	2.25
2.5	1.5
2.75	.75
3	0
...	...
4	0

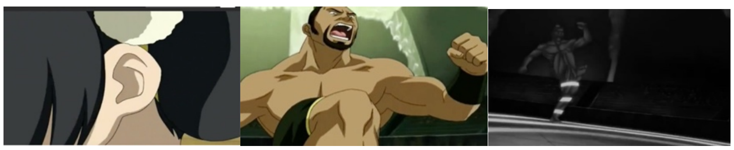
Table 1.2

Overall, the graph and table show us the first move made by the opponent. By tracking the distance of the foot from the ground as time passes, we get a sense of the motion that The Boulder goes through.

Tired of waiting, he lifts his knee powerfully and prepares to drive his heel into the ground, which will probably lead into some devastating earth-bending technique. Toph, however, has other plans, calculating his movements instantly to land her attack with precision.

Creating Toph’s Function

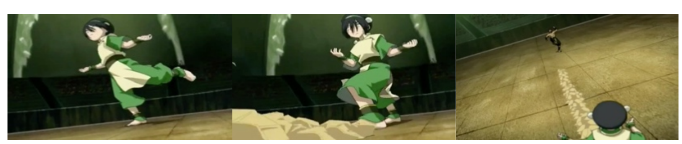
As stated earlier, Aang is looking for an earth-bender who waits and listen, and Toph does just that. In the scene, Toph waits for The Boulder to make the first move, then listens (remember, she’s blind) to adjust accordingly. It looks a little like this:



She hears the earth’s vibrations as he lifts his leg.[11]

Once the vibrations (white lines) reach her, she adjusts her stance accordingly, producing her own vibrations.[12]

And as his foot comes down...[13]



Toph launches her attack, calculated perfectly to catch his foot when it hits the floor.[14]

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[11] Konietzko, Bryan, and Michael Dante DiMartino. “Avatar: The Last Airbender.”
[12] Ibid.
[13] Ibid.
[14] Ibid.

Before actually solving, we need to find a system of equations that will reveal the path of the opponent with the path of Toph. In general, the graph should be somewhat close to:

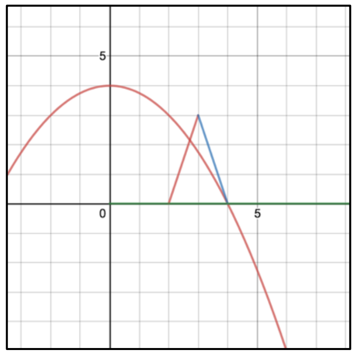


Figure 3

Toph's function, however, will be graphed in different units. Graphing her attack with the distance of her foot from the ground makes a lot less sense, since her attack is not restricted by her physical movements.

The best way to graph Toph's function would be to keep the x as elapsed time, to preserve some consistency, then change the y to the distance between Toph and The Boulder. Consequently, the meaning of the x-axis in this function means that Toph's attack has landed (or met The Boulder). All we have to do is create a realistic function that fits the case.

Again,

$f(x) = ax^2 + bx + c$, where coefficients a, b, and c describe the function

So, it needs to look like the quadratic (in red) drawn earlier, meet the opponent function at the x-axis, and pick up pace towards the end to entail a quick, effective attack.

Just messing with the coefficients, I already notice a couple things.

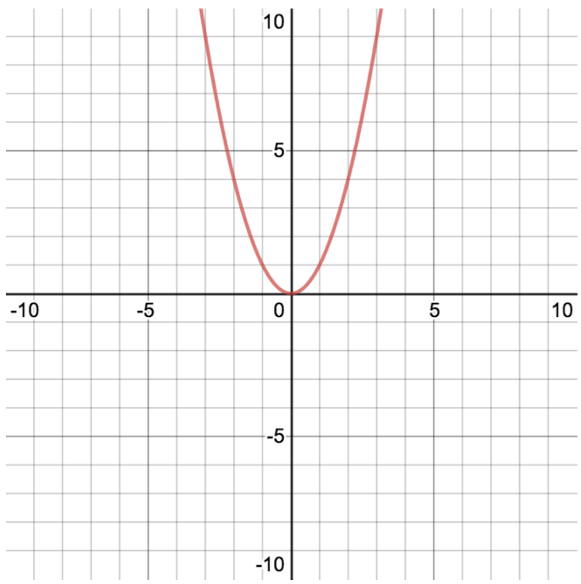


Figure 4.1

$$f(x) = x^2$$

One, the function of x^2 opens upwards. Since we want the graph to move towards the x-axis over time, the a coefficient will have to be negative.

Two, halfway between 0 and the b coefficient is the axis of symmetry, because:

$$x^2 - x$$

After factoring out x, equals

x(x-1)

meaning the two solutions (or x-intercepts) are at x = 0 and x-1 = 0 (or x = 1), giving the axis of symmetry at x = 0.5.

For proof, we can see another example:

$$x^2 - 2x$$

After factoring out x, equals

x(x-2)

which gives the solutions x = 0 and x = 2, wherein there is an axis of symmetry at x = 1. These notes will be helpful in manipulating Toph's function to place it where we want it.

C will also need to be manipulated because Toph's function cannot be at 0 at the y-intersection, or that would mean she's already reached The Boulder before even striking her attack. Further, the y-intercept must be positive, given that she is far from The Boulder and works towards him over time.

To use random coefficients that fulfill our requirements thus far, we now have:

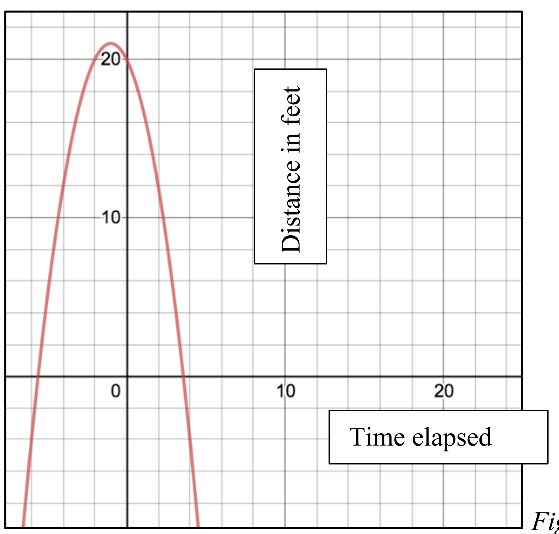


Figure 4.2

$f(x) = x^2 - 2x + 20$

For clarification, the y-axis will be labeled as the distance from The Boulder and x-axis will be labeled in time elapsed. The y-intersection at 20 here means that at the starting point, Toph is 20 feet away from her opponent, which checks out. The x-axis shows us that, with this equation, Toph will reach her opponent somewhere between 0 to 5 seconds.

To find the actual coefficients we need, the information at hand is crucial. Let's start with the fact that our quadratic needs to intersect the x-axis at the point (3, 0).

For the axis of symmetry to be at or before the y-axis, the maximum complimentary solution possible must be at or less than negative 3. If we use the solutions x = 3, -3, we get the function:

$$(x-3)(x+3)$$

FOIL method

$$x^2 + 3x - 3x - 9$$

or

$$f(x) = x^2 - 9$$

Which looks like:

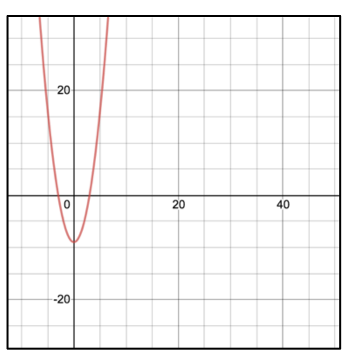


Figure 4.3

Although this graph, at first, looks like it would work, it won't. Before even attempting to flip the a coefficient, I noticed the y-intercept is too low, at a value of 9. We will need something higher, closer to the estimated 20-foot distance estimated to be between the two at the start.

Let's try another value:

$$(x+7)(x-3)$$

Factoring here using FOIL method

$$x^2 - 3x + 7x - 21$$

or

$$f(x) = x^2 + 4x - 21$$

To provide a visual,

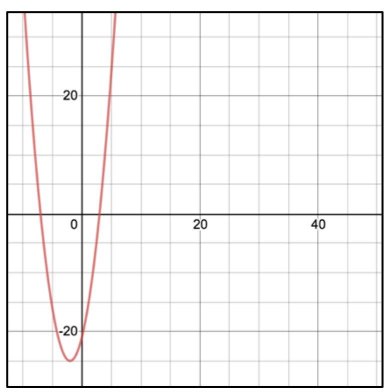


Figure 4.4

Now, this function requires a transformation to make the a-coefficient negative. The rule for reflecting across the x-axis is (x, y) → (x, -y).[15] However, we need the points first. It is given that

[15] Persico, Anthony. "Reflection Over the X and Y Axis: The Complete Guide." *Mashup Math*.

when x is 3, y is 0, just as when x is -7, y is 0. Between 3 and -7 is the axis of symmetry, which represents the function's minimum value at x = -2. To find y, -2 can be plugged in for x:

$$f(x) = x^2 + 4x - 21$$

$$y = (-2)^2 + 4(-2) - 21$$

$$y = 4 - 8 - 21$$

$$y = -25$$

Hence,

(-2, -25).

Now for the reflection, following the rule stated above:

$$(-7, 0) \rightarrow (-7, 0)$$

$$(-2, -25) \rightarrow (-2, 25)$$

$$(3, 0) \rightarrow (3, 0)$$

Which perfectly fits the function,

$$f(x) = -x^2 - 4x + 21$$

(derived from flipping all the signs in the original equation)

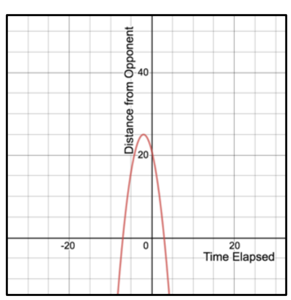
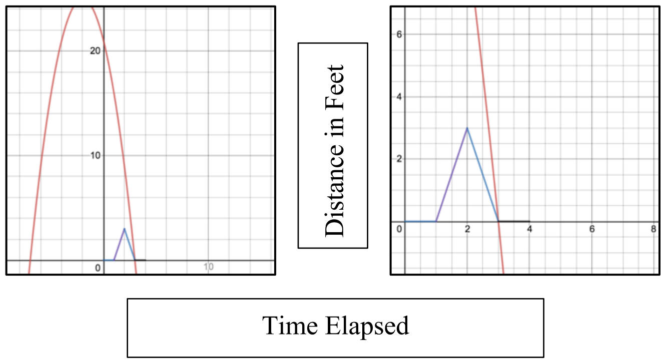


Figure 4.5

with y, in feet, and x, in seconds

Putting It All Together

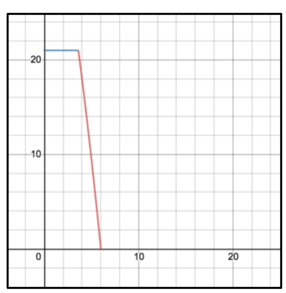
Now that we have the two functions, we can overlap the two.



The final adjustment is to implement a waiting phase at the start because Toph does not initiate her attack immediately, as the graph might assume.

To do this, $f(x) = 21$ will be used in Toph's function from the domain {0 < x < 3}, while the opponent's function will be shifted 3 units to the right.

With the addition to Toph's function, her attack will begin at x = 3 and look like this:

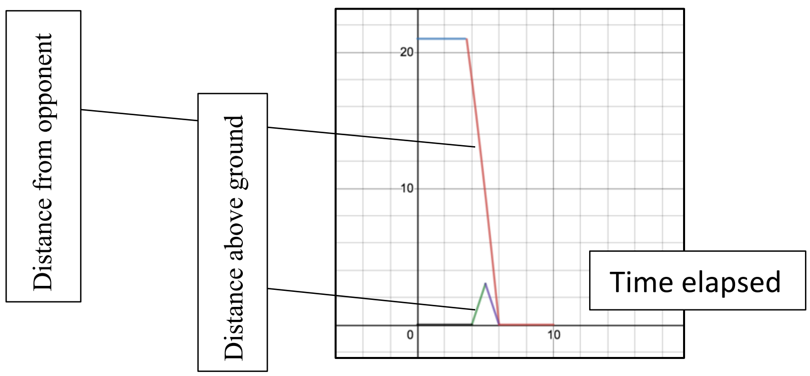


To shift the opponent function, we follow the rule (x, y) → (x + 3, y):

x	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
y	0	.75	1.5	2.25	3	2.25	1.5	.75	0

→

x	4	4.25	4.5	4.75	5	5.25	5.5	5.75	6
y	0	.75	1.5	2.25	3	2.25	1.5	.75	0



One limitation of the final graph I recognize is the fact that using two different y-measures created an awkward assymetrical visual, which makes sense, however, because Toph traveled over twenty feet while The Boulder only moved three.

In summary, the mathematical visual of the battle in Season 2 Episode 6 does nothing short of representing Toph's awesomeness. From Aang's revelation, we see that Toph does indeed "wait and listen," as shown from the first three seconds of the graph. Then, as soon as The Boulder makes his first movement, Toph senses it and attacks.

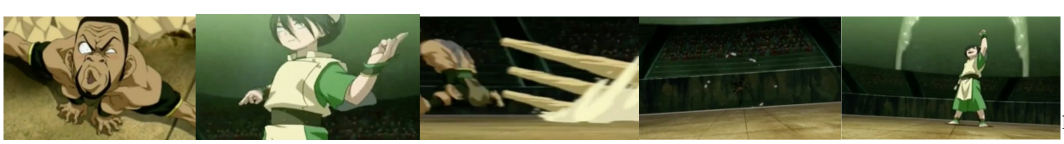
[16]

[16] Konietzko, Bryan, and Michael Dante DiMartino. "Avatar: The Last Airbender."

The graph even shows us the pace of Toph's earth-bending. At an elapsed time of 5 seconds, The Boulder's foot has reached its maximum value, at 3 feet high. Toph, however, is 9 feet away from The Boulder at this exact moment. This means that Toph's earth-bending moves at a pace three times faster than the pace of The Boulder's heel driving into the ground, ensuring the two meet exactly at the same time. When they do meet, shown by the intersection at 6 seconds elapsed, Toph's attack successfully lands.


[17]

Ready to finish her opponent in a humiliating manner, Toph prepares the finishing touch,


[18]

and celebrates her victory, keeping her title as the earth-bending tournament champion.

[17] Konietzko, Bryan, and Michael Dante DiMartino. "Avatar: The Last Airbender."
[18] Ibid.

Works Cited

"Avatar: The Last Airbender." *Metacritic*, 5 May 2006, www.metacritic.com/tv/avatar-the-last-airbender/season-2/episode-6-the-blind-bandit.

Konietzko, Bryan, and Michael Dante DiMartino. "Avatar: The Last Airbender." *Season 2: Episode 6*, Nickelodeon, 2009.

Konietzko, Bryan, et al. *Avatar, the Last Airbender*. Dark Horse Books, 2011.

"The Last Airbender." *IMDb*, IMDb.com, 2010, www.imdb.com/title/tt0938283/plotsummary.

Persico, Anthony. "Reflection Over the X and Y Axis: The Complete Guide." *Mashup Math*, Mashup Math, 10 May 2019, mashupmath.com/blog/reflection-over-x-y-axis.

"Solving and Graphing Quadratic Equations." *American Board for Certification of Teacher Excellence*, www.abcte.org/files/previews/math/s4_p2.html.

"Toph Beifong." *Avatar Wiki*, avatar.fandom.com/wiki/Toph_Beifong.